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SIGNIFICANT PARAMETERS FOR THE EXPANSION OF PROPELLANT GASES IN AN IDEALIZED GUN

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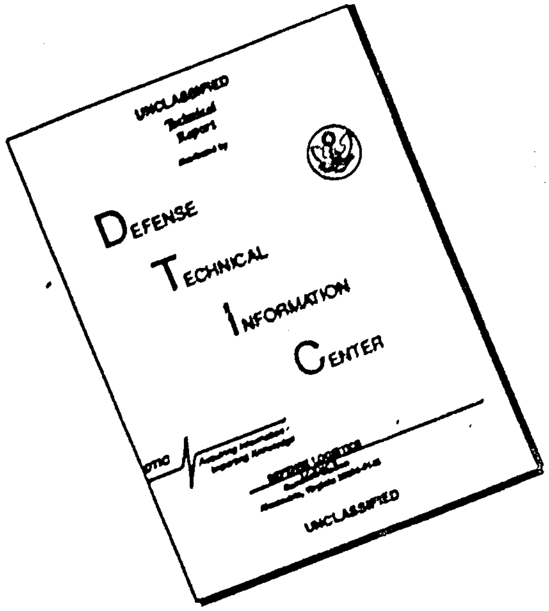

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Aeroballistic Research Report 10

SIGNIFICANT PARAMETERS FOR THE EXPANSION OF PROPELLANT GASES IN AN IDEALIZED GUN

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ABSTRACT: In a previous memorandum (NOLM 10819), a method was developed which permits one to calculate the muzzle velocity and acceleration of a projectile in an idealized gun. In that report it was shown that an accurate method is available and that, given the ratio of charge mass to projectile mass, G/M, the value of the excluded volume in the Abel equation of state, and the initial values of the pressure and temperature, the motion of the projectile within the gun barrel could be calculated for all values of the ratio of specific heats. The present report deals with the problem of simplifying the calculations by reducing the number of parameters to a minimum. It is shown here that for a given gas or gas mixture the problem can be formulated in such a way that the only arbitrary parameter is the ratio G/M. As a result of this investigation it is possible to calculate, by the method developed in NOLM 10819, a single curve giving the velocity of the projectile in the idealized gun barrel for a given value of G/M. This single curve then, by a simple change of scale, yields the velocity of the projectile in the gun for different initial distances between a projectile and breech, for different values of excluded volume, and for different initial pressures and temperatures. The reduction in the calculation labor is significant.

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This report contains information obtained during an investigation of the possibilities of a high-velocity two-stage gun. Earlier phases of the work were reported in Naval Ordnance Laboratory Memorandum 10,81[illegible]. The work was carried out originally under NOL Task Number NOL-159 and later under FR-10 through the support of the Bureau of Ordnance. The report is an interim report; the work is continuing.

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SIGNIFICANT PARAMETERS FOR THE EXPANSION OF PROPELLANT GASES IN AN IDEALIZED GUN

INTRODUCTION

1. In order to examine the possibility of increasing the muzzle velocity of guns, it is necessary that one be able to analyze the processes that go on in the gas behind the missile during the acceleration period in the barrel. Lagrange (reference a) found, many decades ago, that the processes taking place within a gun are simplified if an instantaneous combustion of the powder is assumed so that, before the bullet starts moving, the cylindrical bore in back of it is filled with a hot and highly compressed gas in a uniform thermodynamical state. The motion of the bullet can be correctly described only if the nonsteady changes of state in the expanding gas mass are considered. One method of treating the problem has already been described (reference b); formulas taken from this analysis are reproduced here in brackets.

2. At the back of the moving bullet, tiny expansion wavelets originate at every instant and are sent back through the gas, causing its further expansion. They are reflected at the breech, then travel in a forward direction, reach the bullet, are reflected here, and wander backwards again. This back-and-forth motion continues until the bullet leaves the barrel.

3. The path of every single wavelet is represented, in a (t, x) diagram, by part of one of the characteristic curves associated with the differential equations of the problem. There are two families of such curves composed of infinitely many either "descending" or "ascending" characteristic lines corresponding to the two opposite directions in which the wavelets may move. In actual computation only a finite set of representative curves and wavelets is selected, and the characteristic curves are replaced by polygons, i.e., secant curves.

4. The present investigation was undertaken in order to obtain the maximum information with a minimum of computational effort. It was desirable to so arrange the parameters that a single computation could be used to represent an entire series of computations. The parameters concerned obviously include the initial thermodynamical state of the gas (p_0, ρ_0, T_0), the mass of the bullet, and the mass of the gas. The latter may be expressed by $G = \omega \rho_0 x_0$, introducing two parameters: the cross-sectional area, ω, of the bore and the initial length, x_0, of the gas column. In addition, there is the isentropic exponent, γ, and from the Abel equation of state,

$$p\left(\frac{1}{\rho} - \frac{1}{\beta}\right) = RT, \qquad [1]$$

the excluded volume $1/\beta$, and the gas constant R.

5. In what follows, the equations governing the problem will be written in dimensionless form. It will be shown that the quantities used to make the variables dimensionless can be selected in such a manner that only three parameters are retained in the equations:

$$\gamma, \ \rho_0/\beta, \text{ and } G/M,$$

where M is the mass of the bullet. Of these, γ is unessential if the same powder is employed for different shots. Furthermore, it will be shown that the computation can be arranged so that the position of the bullet becomes independent of the parameter, ρ_0/β, (although the characteristic net does not). This leaves us with only one essential parameter, G/M. Thus, if a single set of curves for various values of G/M is computed, the bullet's path and motion is known for all combinations of other parameters where the same values of G/M enter. The real quantities for a specified problem, of course, are obtained by retransforming the dimensionless ones.

DIMENSIONLESS EQUATIONS

6. In dealing with the Lagrange problem, it is convenient to introduce a quantity, σ, defined by

$$\sigma = \frac{2}{\gamma-1}\,\frac{\beta-\rho}{\beta}\,a, \qquad [6]$$

where "a" is the local accustic speed. The density ρ may be removed from this equation to give a relation between "a" and σ only:

$$a = \frac{\gamma-1}{2}\,\sigma\left[1 + \frac{\rho_0}{\beta-\rho_0}\left(\frac{\sigma}{\sigma_0}\right)^{\frac{2}{\gamma-1}}\right]. \qquad [11]$$

The index "o" refers to the conditions at rest (t = 0). It is seen that σ has the dimension of a velocity.

7. Using optional reference values L and V for length and velocity, we may introduce dimensionless variables (denoted by a bar) as follows:

$$\begin{aligned} x &= L\,\bar{x} \\ u &= V\bar{u}\,, \quad a = V\bar{a}\,, \quad \sigma = V\bar{\sigma} \\ t &= \frac{L}{V}\,\bar{t}\,. \end{aligned} \qquad (1)$$

The quantity u is the local velocity of the gas particles. At the back of the bullet, it is identical with the bullet's speed. Since

$$u_{bullet} = \frac{d}{dt}x_{bullet}$$

the reference value for the time is necessarily L/V. Suitable values for L and V will be given subsequently.

8. The fundamental hydrodynamical equations [10] for the motion of the gas become with [11] and (1):

$$\left.\begin{aligned}
&\frac{\partial \bar{\sigma}}{\partial \bar{t}} + \bar{u}\,\frac{\partial \bar{\sigma}}{\partial \bar{x}} + \frac{\gamma-1}{2}\,\bar{\sigma}\left[1 + \frac{\rho_0}{\beta-\rho_0}\left(\frac{V}{\sigma_0}\right)^{\frac{2}{\gamma-1}}\bar{\sigma}^{\frac{2}{\gamma-1}}\right]\frac{\partial \bar{u}}{\partial \bar{x}} = 0 \\
&\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u}\,\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\gamma-1}{2}\,\bar{\sigma}\left[1 + \frac{\rho_0}{\beta-\rho_0}\left(\frac{V}{\sigma_0}\right)^{\frac{2}{\gamma-1}}\bar{\sigma}^{\frac{2}{\gamma-1}}\right]\frac{\partial \bar{\sigma}}{\partial \bar{x}} = 0 .
\end{aligned}\right\} \qquad (2)$$

They contain the parameters γ , ρ_0/β , V/σ_0 . The boundary condition, $u = 0$, at the breech transforms into $\bar{u} = 0$, where no parameter enters. The boundary condition at the back of the bullet is

$$\frac{du}{dt} = \frac{\omega}{M}\,p_0\left(\frac{\sigma}{\sigma_0}\right)^{\frac{2\gamma}{\gamma-1}} . \qquad [20a]$$

Since relation [21] may be written

$$p_0 = \left(\frac{\gamma-1}{2}\right)^2 \sigma_0^2\,\frac{1}{\gamma}\,\frac{\beta\rho_0}{\beta-\rho_0} , \qquad (3)$$

it follows, by introducing dimensionless variables, that

$$\begin{aligned}
&\frac{d\bar{u}}{d\bar{t}} = B\,\bar{\sigma}^{\frac{2\gamma}{\gamma-1}} , \quad \text{with} \\
&\qquad B = \frac{1}{\gamma}\left(\frac{\gamma-1}{2}\right)^2 \frac{\omega}{M}\,L\,\frac{\beta\rho_0}{\beta-\rho_0}\left(\frac{V}{\sigma_0}\right)^{\frac{2}{\gamma-1}} .
\end{aligned} \qquad (4)$$

9. Even if the parameters appearing in equations (2) and (4) are equal for two expansion processes there are still infinitely many possible solutions for $\bar{\sigma}$, $\bar{u}$ depending on the dimensionless initial length, $\bar{x}_0$, of the gas column, as will be seen at once. The time required for the first expansion wavelet to reach the breech is

$$\bar{t}_{00} = \frac{\bar{x}_0}{\bar{a}_0} \qquad \text{, where from [11] ,}$$

$$\bar{a}_0 = \frac{\gamma-1}{2}\left(\frac{\sigma_0}{V}\right)\frac{\beta}{\beta-\rho_0} .$$

In a (t, x)-diagram the path of the first expansion wavelet is represented by the first descending characteristic line P_0S_{00} (Fig. 1). It intersects the $\bar{t}$-axis at $\bar{t} = \bar{t}_{00}$. Even if σ_0/V and ρ_0/β are kept constant the value of $\bar{t}_{00}$, i.e., the length of the first descending characteristic, varies with $\bar{x}_0$; hence, infinitely many solutions through change of $\bar{x}_0$ alone can be obtained.

10. The first ascending characteristic, $S_{00}P^*$, represents the path of the first expansion wavelet after its reflection at the breech. It

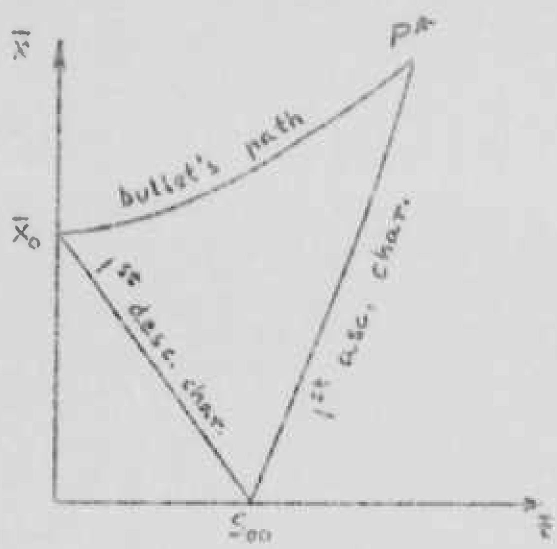

Fig. 1 - Beginning of the Characteristic Net

is evident from Fig. 1 that the location of the point P*, on the bullet's path, will also vary with $\bar{x}_0$. This will be shown mathematically. The expressions [29] and [30] given in reference (b) for the coordinates of P* involve the product λt_{00}, where

$$\left.\begin{aligned} \lambda &= \frac{\gamma+1}{\gamma-1}\,\frac{\omega}{M}\,\frac{p_0}{\sigma_0}\,, \\ t_{00} &= \frac{x_0}{\alpha_0} = \frac{2}{\gamma-1}\,\frac{\beta-\rho_0}{\beta}\,\frac{x_0}{\sigma_0}\,. \end{aligned}\right\}$$

With (3), one obtains from these relations that

$$\lambda t_{00} = \frac{\gamma+1}{2\gamma}\,\frac{G}{M}\,.$$

If we now deal with the distance, $\bar{y} = \bar{x} - \bar{x}_0$, of the bullet from its original seat rather than with that from the breech (distance $\bar{x}$), we find that, at P*:

$$\left.\begin{aligned} \bar{y}^* &= \frac{\gamma-1}{\gamma+1}\,\frac{1}{B}\left(\frac{V}{\sigma_0}\right)^{\frac{2}{\gamma-1}}\left[\frac{\gamma-1}{2} + \left(1+\frac{\gamma+1}{2\gamma}\,\frac{G}{M}\right)^2 - \frac{\gamma+1}{2}\left(1+\frac{\gamma+1}{2\gamma}\,\frac{G}{M}\right)^{\frac{4}{\gamma+1}}\right] \\ \bar{t}^* &= \frac{\gamma-1}{2\gamma}\,\frac{1}{B}\left(\frac{V}{\sigma_0}\right)^{\frac{\gamma+1}{\gamma-1}}\frac{G}{M}\left(2+\frac{\gamma+1}{2\gamma}\,\frac{G}{M}\right) \\ \bar{\sigma}^* &= \frac{\sigma_0}{V}\left(1+\frac{\gamma+1}{2\gamma}\,\frac{G}{M}\right)^{-2\frac{\gamma-1}{\gamma+1}} \\ \bar{u}^* &= \frac{\sigma_0}{V} - \bar{\sigma}^*. \end{aligned}\right\} \quad (5)$$

The last two formulas are derived from [31]. The influence of $\bar{x}_0$ appears in $G = \omega\rho_0\bar{x}_0L$. In addition to G/M the parameters B, σ_0/V, and γ are material to the location of P* and to the values of $\bar{\sigma}$ and $\bar{u}$ at P*. Of these parameters, neither G/M nor γ can be removed from the formulas, whatever values for V and L one may adopt. Hence these two parameters will enter into any computation of Lagrange's problem and cannot be disposed of.

DETERMINATION OF SUITABLE VALUES FOR L AND V

11. Consider now two cases with identical values of γ and G/M. Is it then possible to obtain the first part, P_0P^*, of the bullet's path in a form which is independent of all other parameters? Along that first part

$$\frac{d\tilde{\sigma}}{d\tilde{t}} = -B\,\tilde{\sigma}^{\frac{2\gamma}{\gamma-1}} \qquad (6)$$

Equation (6) is deduced from [20b] and describes the rate of change of the quantity $\tilde{\sigma}$ on $P_o P^*$. It must be solved in such a manner that $\tilde{\sigma} = \tilde{\sigma}_o = {}^{G}/_{V}$ for $\tilde{t} = o$. This initial condition, if the solution is to be parameter-free, requires that V should be proportional to σ_o; choosing the proportionality factor as unity, we have

$$V = \sigma_o \qquad (7)$$

Hence the coefficient B appearing in (6) and (4) is now free of σ_o, but it continues to depend on

$$\frac{\omega \rho_o}{M}\,\frac{\beta}{\beta-\rho_o} = \frac{G}{M}\,\frac{1}{x_o}\,\frac{\beta}{\beta-\rho_o}.$$

However, the reference value L still available for elimination of unwanted parameters, can be chosen so that B becomes a numerical constant, say B = 1:

$$L = \gamma\left(\frac{2}{\gamma-1}\right)^2 \frac{M}{G}\,\frac{\beta-\rho_o}{\beta}\,x_o\,; \quad B = 1 \qquad (8)$$

With the conditions (7) and (8) the solution $\tilde{\sigma} = f(\tilde{t})$ to equation (6) is independent of all parameters excepting γ. The same is true of $\tilde{u}$ (because $\tilde{u} = 1 - \tilde{\sigma}$ on $P_o P^*$) and of $\tilde{y}$ (since $\frac{d\tilde{y}}{d\tilde{t}} = \tilde{u}$ and $\tilde{u} = 0$ for $\tilde{t} = 0$). It is not true of $\tilde{a}$ which is given by

$$\tilde{a} = \frac{\gamma-1}{2}\,\tilde{\sigma}\left[1 + \frac{\rho_o}{\beta-\rho_o}\,\tilde{\sigma}^{\frac{2}{\gamma-1}}\right].$$

It may be noted that, with the aid of (1), (7), (8), equation [24] for the first part, $P_o P^*$, of the path may be written as

$$\tilde{y} = \tilde{t} - \frac{\gamma-1}{2}\left[\left(1 + \frac{\gamma+1}{\gamma-1}\,\tilde{t}\right)^{\frac{2}{\gamma+1}} - 1\right].$$

This relation between the bullet's time- and space-coordinates does indeed depend on γ only. The influence of ${}^{G}/_{M}$ is felt only in that the terminating point P^* and the values of $\tilde{u}$ and $\tilde{\sigma}$ at P^* are different for different values of ${}^{G}/_{M}$ as shown by (5).

12. The general equations (2) and (4) are also simplified by the conditions (7) and (8). However, the parameter ${}^{\rho_o}/_{\beta}$ is not removed from the system (2). Consequently, it also occurs in the pertinent characteristic equations [12a], [12b] which may be written as

$$\frac{d\tilde{y}}{d\tilde{t}} = \tilde{u} - \frac{\gamma-1}{2}\,\tilde{\sigma}\left[1 + \frac{\rho_o}{\beta-\rho_o}\,\tilde{\sigma}^{\frac{2}{\gamma-1}}\right] \qquad (9a)$$

$$\frac{d\tilde{y}}{d\tilde{t}} = \tilde{u} + \frac{\gamma-1}{2}\,\tilde{\sigma}\left[1 + \frac{\rho_o}{\beta-\rho_o}\,\tilde{\sigma}^{\frac{2}{\gamma-1}}\right]. \qquad (9b)$$

The circumflex is used here instead of the bar, in order to indicate clearly that the equations refer to dimensionless quantities along specific curves, namely along characteristic lines.

13. The value of $\bar{x}_0$, i.e., the dimensionless length of the gas column for $\bar{t} = 0$, depends also on ρ_0/β ; from (8):

$$\bar{x}_0 = \frac{X_0}{L} = \frac{1}{\gamma}\left(\frac{\gamma-1}{2}\right)^2 \frac{G}{M} \frac{\beta}{\beta - \rho_0} . \tag{10}$$

In other words, even though the first part of the bullet's path, P_0P^*, can be made independent of ρ_0/β , the characteristic net determined by P_0P^* cannot; the slope of the characteristic lines varies with ρ_0/β and so does the lower boundary $\bar{y} = -\bar{x}_0$. (Compare Fig. 6 at the end of the report.) Since the continuation of the path beyond P^* depends on the continuation of the characteristic net, it appears that beyond the point P^* the path will differ according to which value of ρ_0/β is selected. This is not true, however, as will be shown by the detailed investigation given in the next section.

THE BULLET'S PATH

14. Suppose that some initial part, P_0P_j , of the path (see Fig. 2) and the values of $\bar{u}$ and $\bar{\sigma}$ in that part have been shown to be independent of ρ_0/β . The two different characteristic lines originating for two

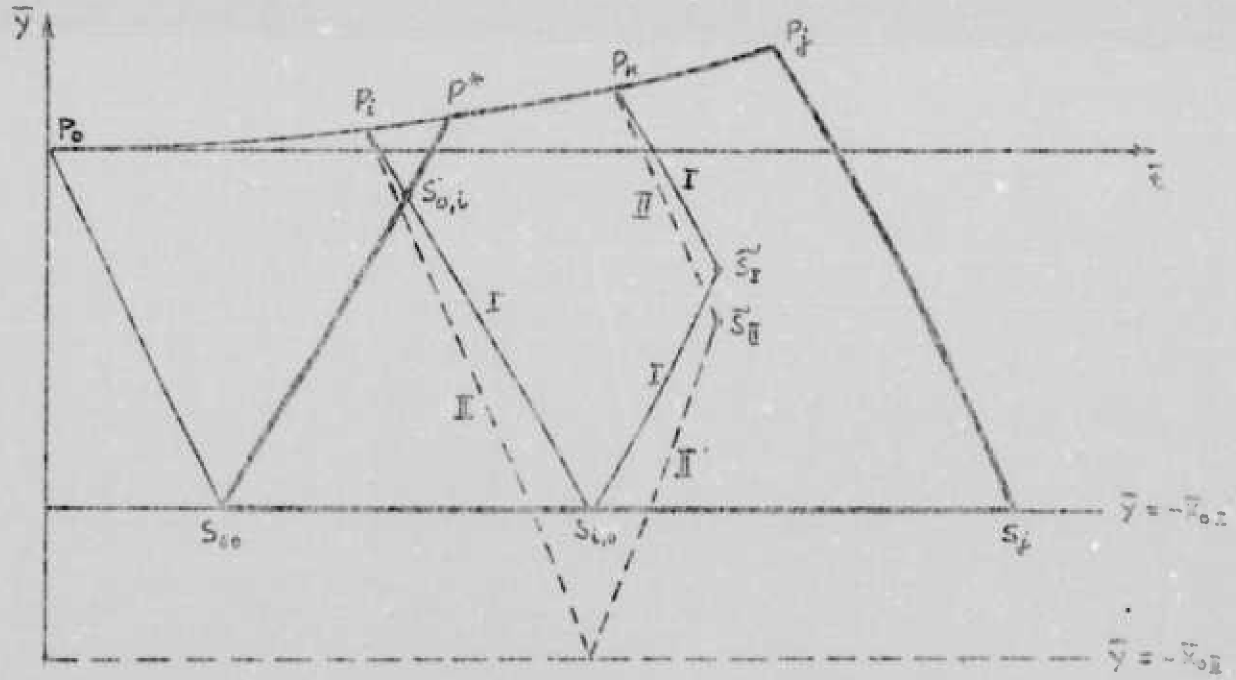

Fig. 2 - Characteristic Lines for Different Values $(\rho_0/\beta)_I$ and $(\rho_0/\beta)_{II}$ and Identical Initial Part P_0P_j

different values of ρ_0/β at any point on P_0P_j will be called corresponding lines; they will be differentiated by the indices I and II. After reflection at the breech they transform into "corresponding" ascending lines I and II (compare Fig. 2).

15. First, the following lemma will be proved:

Any two corresponding characteristic lines intersect at points $\tilde{S}_I$ and $\tilde{S}_{II}$ where

a. the values of the velocities $\tilde{u}$ and $\tilde{\sigma}$ are independent of ρ_0/β ($\tilde{u}_I = \tilde{u}_{II}$, $\tilde{\sigma}_I = \tilde{\sigma}_{II}$; and

b. the abscissa $\tilde{t}$ is independent of ρ_0/β ($\tilde{t}_I = \tilde{t}_{II}$).

16. As an immediate consequence, the rate of change of $\tilde{u}$ and $\tilde{\sigma}$ along one characteristic line is the same as on any corresponding line, i.e., the differential quotients $d\tilde{u}/d\tilde{t}$ and $d\tilde{\sigma}/d\tilde{t}$ along any characteristic line of the domain $P^*P_j S_j S_{00}$ are independent of ρ_0/β (whereas, by (9) the slope $d\tilde{y}/d\tilde{t}$ is not).

17. Part (a) of the lemma is readily demonstrated since on each descending line $\tilde{u} - \tilde{\sigma}$ = const., and on each ascending line $\tilde{u} + \tilde{\sigma}$ = const. This fact (taken from reference b), and the condition $\tilde{u} = 0$ at the breech, yield the relations

$$\tilde{\sigma} = \frac{1}{2}\left(\tilde{\sigma}_i - \tilde{u}_i + \tilde{\sigma}_k - \tilde{u}_k\right),$$

$$\tilde{u} = \frac{1}{2}\left(\tilde{\sigma}_i - \tilde{u}_i - \tilde{\sigma}_k + \tilde{u}_k\right),$$

where the indices refer to P_i and P_k on P_0P_j (Fig. 2). According to assumption, the right-hand sides, and hence $\tilde{\sigma}$ and $\tilde{u}$, are independent of ρ_0/β ; or $\tilde{\sigma}_I = \tilde{\sigma}_{II}$, $\tilde{u}_I = \tilde{u}_{II}$.

18. Part (b) is much more difficult to prove.

19. All the points, $\tilde{S}$, in question are located within and on the boundary of the region $P^*P_j S_j S_{00}$ of Fig. 2. The abscissa $\tilde{t}$ does not depend on ρ_0/β along the portion, $S_{00}P^*P_j$, of the boundary. This is true on P^*P_j by assumption again. On $S_{00}P^*$ the abscissa, $\tilde{t}_{0,i}$, is given by

$$1 + \frac{\gamma+1}{\gamma-1}\tilde{t}_{0,i} = \left(1 + \frac{\gamma+1}{2\gamma}\frac{C}{M}\right)\left(\frac{1}{\tilde{\sigma}_{0,i}}\right)^{\frac{1}{2}\frac{\gamma+1}{\gamma-1}} \qquad (11)$$

It has been pointed out in reference (b) that, at $S_{0,i}$ (Fig. 2), the value of the quantity $\tilde{\sigma}$ is identical with its value at P_i. Thus since it does not vary with ρ_0/β , the abscissa $\tilde{t}_{0,i}$ does not vary either. Relation (11) follows from [28] with the aid of (1), (3), (5), (7), and (8).

20. The same inference, however, cannot be immediately drawn regarding the values of $\bar{t}$ on the remaining portion of the boundary or in the interior of the domain $P^*P_jS_jS_{oo}$. For a conclusive demonstration we will have to investigate in detail the variation of $\bar{t}$ by setting up and discussing a differential equation for that variable. The quantities $\bar{u}$ and $\bar{\sigma}$ will be chosen as independent variables in this equation. The mathematics will be given in the next section; the results may be briefly summed up as follows.

21. The domain $P^*P_jS_jS_{oo}$ is mapped onto a domain $\Pi^*\Pi_j\Sigma_j\Sigma_{oo}$ in a $(\bar{u},\bar{\sigma})$ -plane (Fig. 3). The variable $\bar{t}$ satisfies the differential equation

$$\frac{\partial^2 \bar{t}}{\partial \bar{u}^2} - \frac{\partial^2 \bar{t}}{\partial \bar{\sigma}^2} - \frac{\gamma+1}{\gamma-1}\,\frac{1}{\bar{\sigma}}\,\frac{\partial \bar{t}}{\partial \bar{\sigma}} = 0. \quad (12)$$

Obviously, this equation does not vary with ρ_o/β, neither does the boundary

$$\Pi^*\Pi_j\Sigma_j\Sigma_{oo},$$

nor do the conditions imposed there on the function $\bar{t}$, as will be shown in the next section. The solution of (12) then is necessarily independent of ρ_o/β.

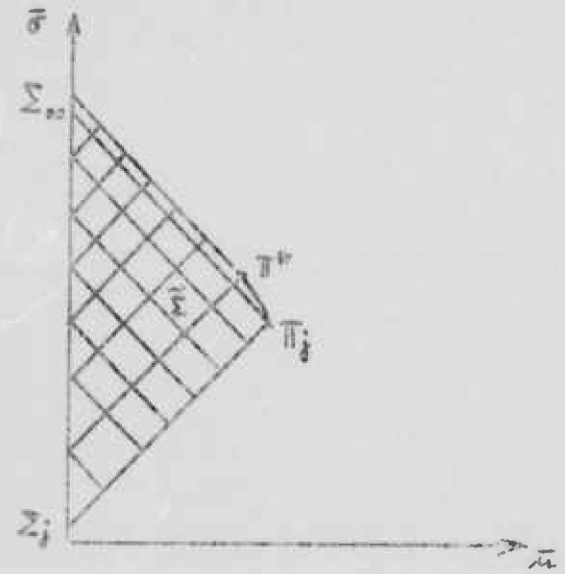

Fig. 3 - The $(\bar{u},\bar{\sigma})$ -plane

22. The two "corresponding" points $\tilde{S}_I$ and $\tilde{S}_{II}$ (Fig. 2) may be viewed as specimens of a set of infinitely many such points differentiated by the value of ρ_o/β and each determined by a pair of characteristics originating at P_j and P_k. Despite the difference in ρ_o/β the values of $\tilde{u}$ and $\tilde{\sigma}$ at all these points are the same, and all are therefore mapped into the same point $\tilde{\Sigma}$ (Fig. 3). If the value of $\bar{t}$ at this point varied with ρ_o/β the points $\tilde{S}$ would have different abscissas; since it does not, all these abscissas are identical, as stated in part (b) of the lemma.

23. Consider now, in particular, some point $\tilde{S}(\tilde{t},\tilde{y})$ on the descending characteristic line P_jS_j (Fig. 2). The path, if continued beyond P_j, and the ascending characteristic through $\tilde{S}$ may intersect at the point $P(\bar{t}\ \bar{y})$, where the dimensionless velocities are $\bar{\sigma}$ and $\bar{u}$ (Fig. 4). The abscissa, $\tilde{t}$, of $\tilde{S}$ will be chosen as the independent variable. The rate of change, in terms of $\tilde{t}$, of the variables $\bar{t}$, $\bar{y}$, $\bar{u}$, $\bar{\sigma}$ will be investigated in the vicinity of P_j. Should it prove to be independent of ρ_o/β, then the location of P, if visualized as the immediate successor of P_j on the path, would be the same for all values of ρ_o/β. Likewise the quantities $\bar{u}$ and $\bar{\sigma}$ at P would not depend on that parameter.

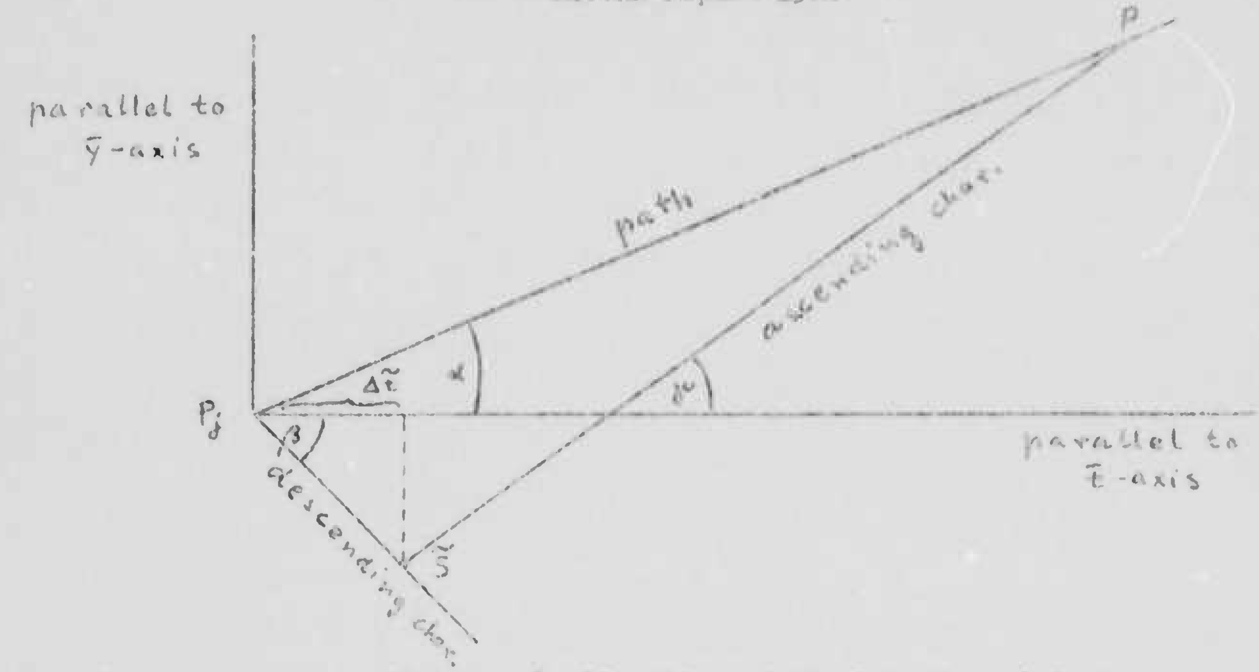

Fig. 4 - Continuation of the Path Beyond P_j

24. The sides of the triangle $P_j \tilde{\xi} P$ (Fig. 4) can be taken as straight lines if $\Delta\tilde{t} = \tilde{t} - \tilde{t}_j$ is sufficiently small. It then follows from the geometry of Fig. 4 that

$$\bar{t} - \bar{t}_j = \Delta\bar{t} = \frac{\tan\gamma + \tan\beta}{\tan\gamma - \tan\alpha}\,\Delta\tilde{t}.$$

When $\Delta\tilde{t} \to 0$, we have

$$\tan\alpha \to \bar{u}_j\,, \quad \tan\beta \to \bar{a}_j - \bar{u}_j\,, \quad \tan\gamma \to \bar{a}_j + \bar{u}_j\,.$$

In other words, the slopes of the path and of the characteristic lines approach the values that prevail at P_j. It follows that

$$\lim_{\Delta\tilde{t} \to 0} \frac{\Delta\bar{t}}{\Delta\tilde{t}} = \left(\frac{d\bar{t}}{d\tilde{t}}\right)_{P_j} = 2\,.$$

It is remarkable that the quantity $\bar{\Xi}_j$ has canceled out, i.e., the influence of ρ_0/ρ (formula [11]) has disappeared.

25. As to the rate of change of $\bar{y}$ and $\bar{u}$ in the vicinity of P_j we see that

$$\left(\frac{d\bar{y}}{d\tilde{t}}\right)_{P_j} = \left(\frac{d\bar{y}}{d\bar{t}}\,\frac{d\bar{t}}{d\tilde{t}}\right)_{P_j} = 2\bar{u}_j$$

and, owing to (4) and (8), that

$$\left(\frac{d\bar{u}}{d\tilde{t}}\right)_{P_j} = \left(\frac{d\bar{u}}{d\bar{t}}\,\frac{d\bar{t}}{d\tilde{t}}\right)_{P_j} = 2\bar{\sigma}_j^{\frac{2\kappa}{\kappa-1}}.$$

Both right-hand sides, according to assumption, do not change with ρ_0/β.

26. In order to determine $\left(\frac{d\bar{\sigma}}{d\bar{t}}\right)_{P_j}$ two facts must be linked together.

a. Since $\tilde{t}$, $\tilde{u}$, $\tilde{\sigma}$ denote dimensionless values along the characteristic P_jS_j the above lemma can be applied, i.e., the differential quotients $d\tilde{u}/d\tilde{t}$ and $d\tilde{\sigma}/d\tilde{t}$ are independent of ρ_0/β.

b. From reference (b), the values $\tilde{u}$, $\tilde{\sigma}$ at $\tilde{S}$ and $\bar{u}$, $\bar{\sigma}$ at P are interconnected by

$$\tilde{u} + \tilde{\sigma} = \bar{u} + \bar{\sigma} .$$

27. It follows that

$$\left(\frac{d\bar{\sigma}}{d\bar{t}}\right)_{P_j} = \left(\frac{d\tilde{u}}{d\tilde{t}} + \frac{d\tilde{\sigma}}{d\tilde{t}} - \frac{d\bar{u}}{d\bar{t}}\right)_{P_j},$$

where the right side is not affected by changes of ρ_0/β.

28. From these results the important inference can be drawn that if some initial portion P_0P_j of the path and the values of $\bar{u}$ and $\bar{\sigma}$ on it are independent of ρ_0/β, it is also true for the immediate successor P of P_j on the path. The same reasoning then can be applied to the point following the successor, and so forth; in short, the entire path does not vary with ρ_0/β. As an initial portion we may take the arc P_0P^* for which independency of ρ_0/β can be attained, as shown in the third section.

29. Since one is chiefly interested in the bullet's position and velocity, a considerable amount of computing labor can be saved by simply putting $\rho_0/\beta = 0$ i.e., $\beta = \infty$. For then the ideal equation of state is substituted for [1], and the unwieldy relation [11] which is used at every computational step for determining the local acoustic speed is simplified into $\tilde{a} = \frac{\gamma-1}{2}\tilde{\sigma}$. The characteristic net of course will develop differently than it would with the correct value of ρ_0/β. (Compare Fig. 6 at the end of the report.) However, the dimensionless path will be the same, and so will the bullet's dimensionless velocity. The dimensional values for position and velocity will be found from (1), inserting there the quantities L and V as defined by (8) and (7). In (8) the correct value of ρ_0/β must be employed.

30. It is of interest to note that the important result derived in this section for the exact solution of Lagrange's problem has been found also to be true for the so-called Piddock-Kent special solution. (Compare reference (c), page 7.)

THE EQUATION FOR t

31. The equation (12) for $\bar{t}$ holds also for dimensional variables. We therefore proceed to derive it for t instead of $\bar{t}$, with u and σ as independent variables. The formal relations

$$u_x = D t_\sigma \ , \quad u_t = -D x_\sigma \ , \quad \sigma_x = -D t_u \ , \quad \sigma_t = D x_u \tag{13}$$

assume that

$$D = \begin{vmatrix} u_x & u_t \\ \sigma_x & \sigma_t \end{vmatrix} \neq 0$$

and transform the system [10] : $\sigma_t + u\sigma_x + a u_x = 0$; $u_t + u u_x + a \sigma_x = 0$ into

$$\left. \begin{aligned} x_u &= u t_u - a t_\sigma \\ x_\sigma &= -a t_u + u t_\sigma . \end{aligned} \right\} \tag{14}$$

If the first and the second of these equations are partially differentiated with respect to σ and u, respectively, we obtain:

$$-\frac{da}{d\sigma} t_\sigma - a t_{\sigma\sigma} = -a t_{uu} + t_\sigma .$$

Since by [11]

$$\frac{da}{d\sigma} = \frac{\gamma+1}{\gamma-1} \frac{a}{\sigma} - 1$$

the relation just derived may be written

$$t_{uu} - t_{\sigma\sigma} - \frac{\gamma+1}{\gamma-1} \frac{t_\sigma}{\sigma} = 0 . \tag{15}$$

This is equation (12) in dimensional variables. It is independent of any parameter except γ . The characteristic curves associated with (15) are given by the equations

$$\sigma + u = \text{const.} , \qquad \sigma - u = \text{const.} \tag{16}$$

These straight lines form a rectangular system making 45-degree angles with the axes u = 0 and σ = 0. This is indicated on Fig. 3.

32. Returning now to dimensionless variables we see at once that $S_{00}P^*$ (Fig. 2) corresponds to $\Sigma_{00}\Pi^*$ (Fig. 3), this segment being part of the characteristic line $\bar{u} + \bar{\sigma} = 1$, a condition satisfied on $S_{00}P^*$. There is no dependency on σ_0/β here. The values $\bar{u}^*$, $\bar{\sigma}^*$ at P^*, i.e., the coordinates of Π^*, by force of assumption are also independent of σ_0/β , and so are the coordinates of all the points on the arc $\Pi^* \Pi_j$. This is especially true for the coordinates $\bar{u}_j$, $\bar{\sigma}_j$ of Π_j and, therefore, for the characteristic line $\bar{u} - \bar{\sigma} = \bar{u}_j - \bar{\sigma}_j$ = const. The segment $\Pi_j \Sigma$.

of that line is the image of P_jS_j. Thus we see that the image $\Sigma_{00}\Pi^*\Pi_j\Sigma_j$ of the domain $S_{00}P^*P_jS_j$ remains the same whatever value for ρ_0/β may be selected.

33. The boundary conditions imposed on the solution to (12) remain unchanged, too. There is no such condition along $\Pi_j\Sigma_j$. On $\Sigma_{00}\Pi_j$, however, we must prescribe the values of $\bar{t}$ on $S_{00}P_j$, i.e., the values of the abscissas of the points on $S_{00}P_j$. On $S_{00}P^*$, these are given by $\bar{\tau}_{0i}$ (equation (11)) and do not vary with ρ_0/β. On P^*P_j the abscissas are independent of ρ_0/β simply by assumption. Finally, the straight-line segment $\Sigma_{00}\Sigma_j$ is the image on the line $S_{00}S_j$ along which $\bar{u} = 0$. This condition necessitates that

$$\frac{\partial \bar{u}}{\partial \bar{t}} = 0 \qquad \text{on } S_{00}S_j .$$

Then, from the second equation (2)

$$\frac{\partial \bar{\sigma}}{\partial \bar{x}} = 0 \qquad \text{on } S_{00}S_j$$

This transforms, by (13), into

$$\frac{\partial \bar{t}}{\partial \bar{u}} = 0 \qquad \text{on } \bar{u} = 0 \text{, i.e., on } \Sigma_{00}\Sigma_j .$$

Again there is no dependency on ρ_0/β .

34. With this, all statements used but not proved in the previous section have been shown to be true.

CONCLUDING REMARKS

35. We have seen that, as far as the bullet's motion is concerned, only two parameters essential to Lagrange's problem are left: the ratio of specific heats (γ) and the mass ratio ($\frac{C}{M}$). Further, in dimensionless variables the bullet's position and velocity can be found by using the perfect gas law instead of [1]. The imperfectness of the gas is of influence when, by (7) and (8), dimensional variables are determined from the dimensionless ones.

36. Since by [6] the reference value V may be written

$$V = \frac{2}{\gamma-1}\,\frac{\beta-\rho_0}{\beta}\,a_0$$

it is immediately seen that any velocity $u = \bar{u}\,V$ will increase with the initial value of the acoustic speed. This is true, in particular, for the muzzle velocity. A rearrangement by [3] and [1] gives

$$V = \frac{2}{\gamma-1}\sqrt{\gamma R T_0}$$

Other things being equal, the gas with the higher specific gas constant, i.e., with the lower molecular weight, will provide the higher muzzle velocity. It will also require the greater barrel length, unless the initial pressure is raised. This can be seen from formula (8) which may be written

$$L = \gamma \left(\frac{2}{\gamma-1}\right)^2 \frac{M}{\omega} \frac{\beta-\rho_0}{\beta\rho_0} = \gamma \left(\frac{2}{\gamma-1}\right)^2 \frac{M}{\omega} \frac{R T_0}{p_1}$$

For two gases with the specific gas constants R_1 and R_2 the muzzle velocity is in the ratio $\sqrt{R_1} : \sqrt{R_2}$ whereas the barrel length, $y_{muzzle} = \bar{y}_{muzzle} \times L$ is in the ratio $R_1 : R_2$ (provided that $\gamma, G, M, \omega, T_0, p_0$ are identical in the two cases).

37. For a survey with a given value of γ the only essential parameter is $\frac{G}{M}$. If we first take the value $\frac{G}{M} = \infty$ i.e., $x_0 = \infty$, no reflected wavelet will be sent back to the bullet. Plotting the dimensionless velocity of the bullet against its dimensionless location, we will obtain the main curve or Fig. 5. For finite values of $\frac{G}{M}$ the dimensionless distance, given by (10), of the breech from the original seat of the bullet will be finite too. However, the $(\bar{y}, \bar{u})$-curve will not change before the first reflected wavelet arrives at the bullet. The pertinent values of $\bar{y}$ and $\bar{u}$ can be found from the set of formulas on page 4; they are

$$\bar{y}^* = \frac{\gamma-1}{\gamma+1}\left[\frac{\gamma-1}{2} + k^2 - \frac{\gamma+1}{2} k^{\frac{4}{\gamma+1}}\right]$$

$$\bar{u}^* = 1 - k^{-2\frac{\gamma-1}{\gamma+1}}$$

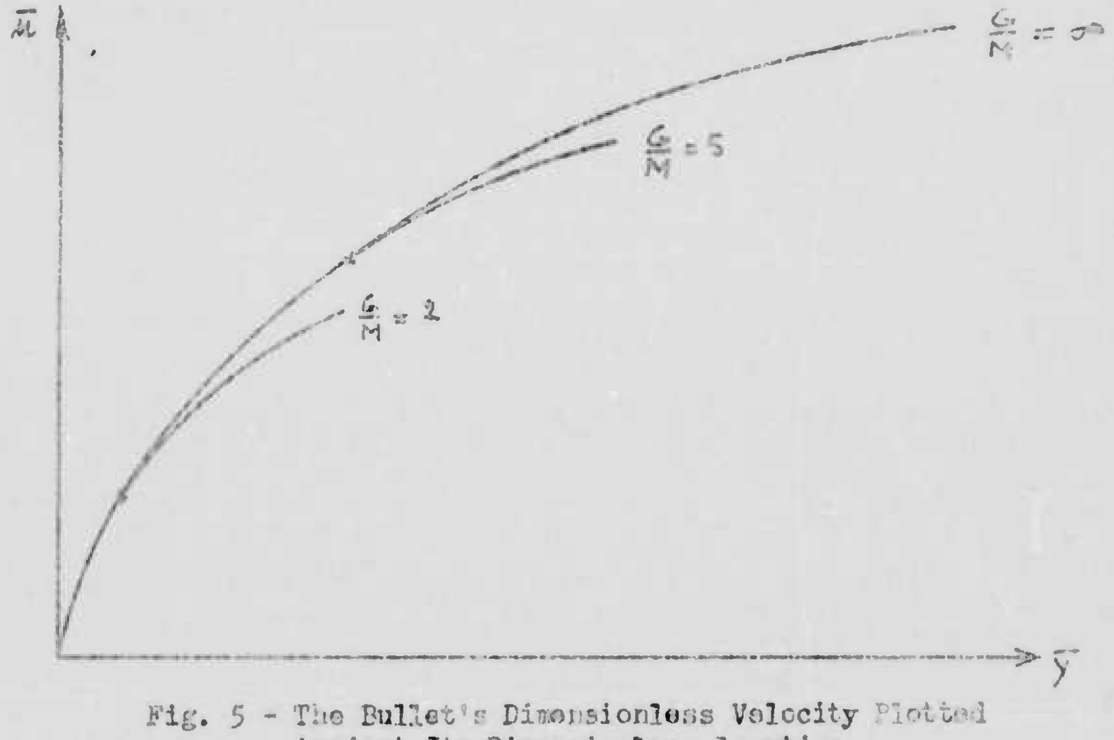

Fig. 5 - The Bullet's Dimensionless Velocity Plotted Against Its Dimensionless Location

where

$$\kappa = \left(1 + \frac{\gamma+1}{2\gamma}\,\frac{G}{M}\right)$$

It is easily seen that both $\bar{\varsigma}^*$ and $\bar{u}^*$ increase with κ, i.e., with $\frac{G}{M}$. Hence the smaller $\frac{G}{M}$, the sooner the pertinent curve branches off the main curve on Fig. 5.

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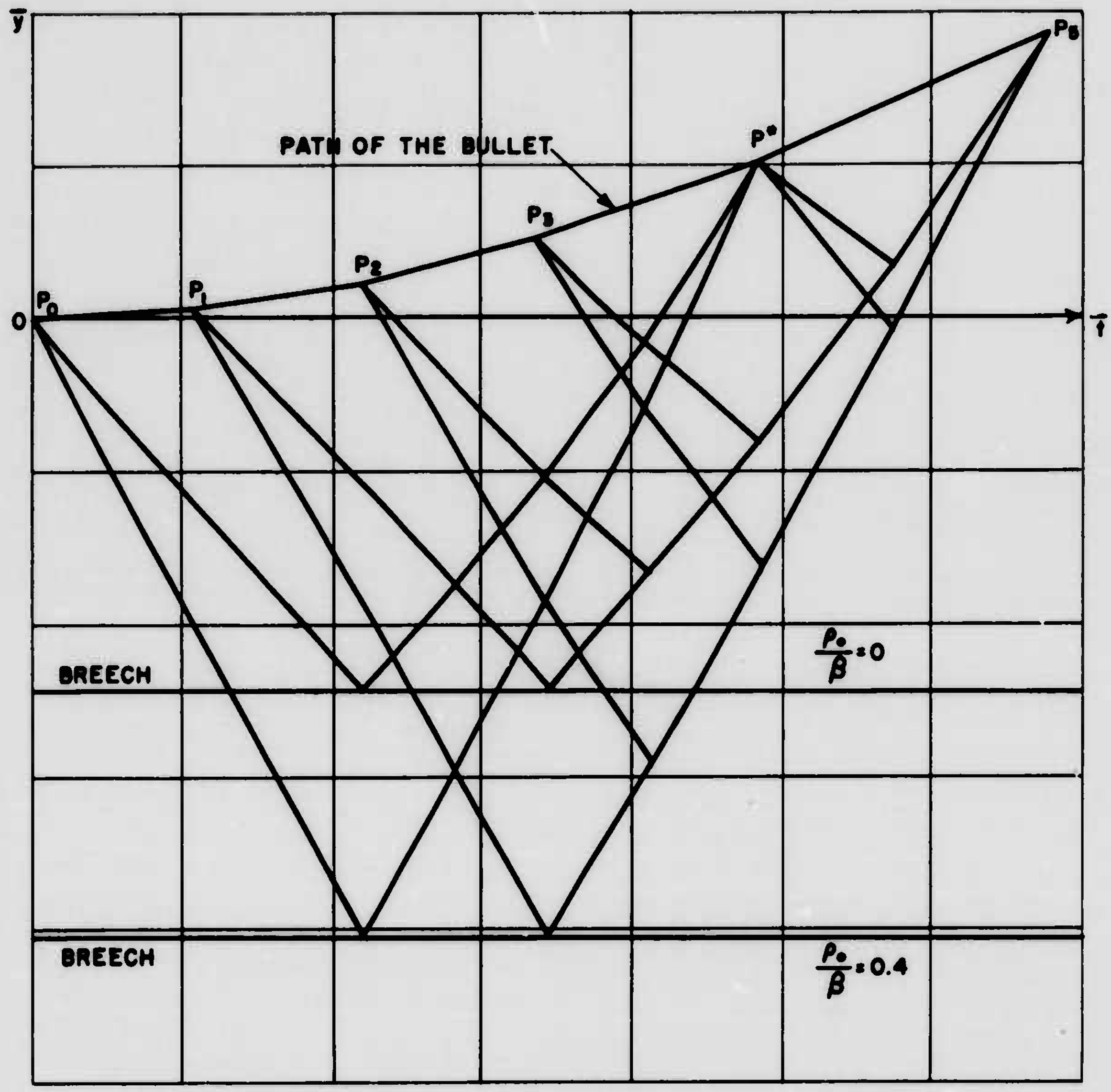

FIG.6 TWO CASES WITH IDENTICAL PATH BUT DIFFERENT CHARACTERISTIC NET $\left(\frac{G}{M} = 0.24,\ \gamma = \frac{11}{9} \right)$